

ELLIPTIC AND PARABOLIC EQUATIONS WITH DIRICHLET CONDITIONS AT INFINITY ON RIEMANNIAN MANIFOLDS

P. MASTROLIA¹, D. D. MONTICELLI² AND F. PUNZO³

ABSTRACT. We investigate existence and uniqueness of bounded solutions of parabolic equations with unbounded coefficients in $M \times \mathbb{R}_+$, where M is a complete noncompact Riemannian manifold. Under specific assumptions, we establish existence of solutions satisfying prescribed conditions at infinity, depending on the direction along which infinity is approached. Moreover, the large-time behavior of such solutions is studied. We consider also elliptic equations on M with similar conditions at infinity.

1. INTRODUCTION

We are concerned with bounded solutions of linear elliptic equations of the type

$$(1.1) \quad a\Delta u + cu = f \quad \text{in } M,$$

where M is a complete, m -dimensional, noncompact Riemannian manifold with metric g , Δ is the Laplace-Beltrami operator with respect to g ; furthermore, $a > 0$, $c \leq 0$, $c, f \in L^\infty(M)$. Observe that, at *infinity*, the function a can be unbounded, or it can tend to 0, or it needs not to have a limit.

Moreover, we study bounded solutions of linear parabolic Cauchy problems of the following form

$$(1.2) \quad \begin{cases} \partial_t u = a\Delta u + cu + f & \text{in } S := M \times (0, \infty), \\ u = u_0 & \text{in } M \times \{0\}, \end{cases}$$

where $u_0 \in L^\infty(M)$. Precise assumptions on a, c, f , and u_0 will be made in Section 3 below.

Existence and uniqueness of solutions of elliptic equations and of parabolic problems have been largely investigated, in the case $M = \mathbb{R}^m$ (see e.g. [4], [13], [14], [15], [16], [20], [21], [22], [23]). In particular, in [15, 16] for suitable classes of elliptic and parabolic equations, it is shown that it is possible to prescribe Dirichlet type conditions at *infinity*. More precisely, one can impose that the solutions at infinity, along radial directions, approach any given continuous function defined on the unit sphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$. It is also observed that in \mathbb{R}^m such results cannot hold in general for operators of the form appearing in equation (1.1) or in problem (1.2).

The situation is quite different on negatively curved Riemannian manifolds. In fact, in [2, Theorem 3.2] (see also [2, 3, 5, 27]) it is shown that if M is a complete, simply connected Riemannian manifold with sectional curvatures bounded between two negative constants, then for every continuous function γ on the sphere at infinity $S_\infty(M)$ there exists a unique solution u of equation

$$(1.3) \quad \Delta u = 0 \quad \text{in } M,$$

Date: December 1, 2015.

2010 *Mathematics Subject Classification.* 35J25, 35J67, 35K10, 35K20, 58J05, 58J32, 58J35.

Key words and phrases. Elliptic equations, parabolic equations, Dirichlet conditions at infinity, Riemannian manifolds.

¹Università degli Studi di Milano, Italy. Email: paolo.mastrolia@gmail.com.

²Politecnico di Milano, Italy. Email: dario.monticelli@polimi.it.

³Università della Calabria, Italy. Email: fabio.punzo@unical.it.

P. Mastrolia, D.D. Monticelli and F. Punzo are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The research that led to the present paper was partially supported by the grant "Analisi Globale, PDE's e Strutture Solitoniche" of the group GNAMPA of INdAM.

that is equation (1.1) with $c = f = 0$, such that $u = \gamma$ on $S_\infty(M)$. In this kind of results the Martin boundary plays a prominent role, and theoretical potential theory is heavily exploited. Indeed, more general elliptic equations are considered, but always without any external forcing term f and zero order term c . Note that the presence of a zero order term in equation (1.1) may remarkably alter the situation. Indeed, let $\lambda > 0$ be a constant; from [7, Theorem 6.2] it follows that equation

$$(1.4) \quad \Delta u - \lambda u = 0 \quad \text{in } M,$$

that is equation (1.1) with $f = 0$ and $c = -\lambda$, admits a unique bounded solution, if M is the hyperbolic space \mathbb{H}^m . Hence the result in [2], which we recalled above, does not hold for equation (1.4), as it would imply nonuniqueness of bounded solutions of equation (1.4).

Some general results concerning conditions at infinity for solutions of parabolic equations on Riemannian manifold are established in [20, 21]. The Martin boundary is used; moreover, a representation formula is derived for positive solutions of the Cauchy problem, associated to divergence form elliptic operators.

In this paper, under suitable assumptions on a and M (see **(HP1)** below), we prove existence of solutions of the elliptic equation (1.1) satisfying prescribed conditions at infinity. More precisely, consider on M the polar coordinates $(r, \theta) \in (0, \infty) \times \mathbb{S}^{m-1}$, with respect to some fixed origin $o \in M$; here $\mathbb{S}^{m-1} := \{x \in \mathbb{R}^m : |x| = 1\}$, see also Section 2. Define

$$(1.5) \quad \mathcal{A} := \left\{ f \in C^\infty((0, \infty)) \cap C^1([0, \infty)) : f'(0) = 1, f(0) = 0, f > 0 \text{ in } (0, \infty) \right\}.$$

We always make the following assumption:

(HP1). (i) There exist a point $o \in M$ with $\text{Cut}(o) = \emptyset$, i.e. with empty *cut-locus*, a constant $R_0 > 0$ and a function $\underline{a} \in C([R_0, \infty))$ such that

$$a(x) \geq \underline{a}(r) > 0 \quad \text{for all } x \in M \text{ with } r = \text{dist}(x, o) \geq R_0;$$

(ii) there exists a function $\psi \in \mathcal{A}$ such that

$$K_\omega(x) \leq -\frac{\psi''(r)}{\psi(r)} \quad \text{for all } x = (r, \theta) \in M \setminus \{o\},$$

$$\int_1^\infty \frac{dr}{\psi^{m-1}(r)} < \infty, \quad \int_1^\infty \left(\int_r^\infty \frac{d\xi}{\psi^{m-1}(\xi)} \right) \frac{\psi^{m-1}(r)}{\underline{a}(r)} dr < \infty.$$

Here $K_\omega(x)$ denotes the radial sectional curvature at x (see Section 2).

From **(HP1)** it follows that in our result there is an interplay between the coefficient $a(x)$ and the manifold M , through the function ψ which is in turn related to the radial sectional curvature. Observe that if $a(x) \equiv 1$, then condition **(HP1)** implies that M is stochastically incomplete (see e.g. [7]). Moreover, it is direct to see that if **(HP1)** holds, then M is non-parabolic, i.e. it admits a positive Green function $G(x, y) < \infty$ for every $x, y \in M$, $x \neq y$; indeed, by [8, Theorem 4.2],

$$G(x, o) \leq \tilde{C} \int_{r(x)}^\infty \frac{d\xi}{\psi^{m-1}(\xi)} \quad (x \in M \setminus \{o\}).$$

Thus, we also have that for some compact subset $K \subset M$,

$$\int_{M \setminus K} \frac{G(x, o)}{a(x)} d\mu(x) < \infty.$$

Under suitable additional hypotheses on the coefficient $a(x)$ (see conditions **(HP0)** and (3.2) below), we show that, for any $\gamma \in C(\mathbb{S}^{m-1})$, there exists a unique solution of the elliptic equation (1.1) satisfying

$$(1.6) \quad \lim_{r \rightarrow \infty} u(r, \theta) = \gamma(\theta) \quad \text{uniformly w.r.t. } \theta \in \mathbb{S}^{m-1}.$$

Note that condition (1.6) can be regarded as a Dirichlet condition at infinity, depending on the direction along which infinity is approached.

Moreover, for any given function $\tilde{\gamma} \in C(\mathbb{S}^{m-1} \times [0, \infty))$, we prove that there exists a unique solution of problem (1.2) such that

$$(1.7) \quad \text{for each } T > 0, \lim_{r \rightarrow \infty} u(r, \theta, t) = \tilde{\gamma}(\theta, t) \quad \text{uniformly w.r.t. } \theta \in \mathbb{S}^{m-1}, t \in [0, T],$$

provided

$$(1.8) \quad \lim_{r \rightarrow \infty} u_0(r, \theta) = \tilde{\gamma}(\theta, 0) \quad \text{uniformly w.r.t. } \theta \in \mathbb{S}^{m-1}.$$

Note again that condition (1.7) can be regarded as a time-dependent Dirichlet condition at infinity, depending on the direction along which infinity is approached.

We should note that when $\psi(r) = r$, and thus $M = \mathbb{R}^m$, our result cannot be applied (see Remark 3.6 below); this is in accordance with remarks made above (see [15]). On the other hand, we want to stress that our results are completely new also for problem

$$(1.9) \quad \begin{cases} \partial_t u = \Delta u & \text{in } S \\ u = u_0 & \text{in } M \times \{0\}, \end{cases}$$

i.e. problem (1.2) with $a \equiv 1, c \equiv f \equiv 0$.

In order to obtain existence of solutions to problem (1.1) satisfying (1.6), we construct and use suitable *barrier functions* at infinity (see Section 4 below). Furthermore, by means of such barriers, we construct convenient subsolutions and supersolutions, also depending on the time variable t , in order to prescribe condition (1.7) for solutions of problem (1.2). We explicitly note that in order to construct such barriers a prominent role is played by **(HP1)**. However, the same existence results that we prove hold also on more general Riemannian manifolds, if one a priori assumes the existence of such barriers.

A similar approach has been used in [5], where barriers which indeed are subharmonic functions have been exploited. In fact, in [5] it has been shown the existence of solutions satisfying Dirichlet conditions at infinity only for equation (1.3); moreover, it is supposed that M is a spherically symmetric manifold with negative radial sectional curvature satisfying a suitable bound from above (see Section 2).

The paper is organized as follows. In Section 2 we introduce some basic notions and tools from Riemannian geometry, while in Section 3 we state our main results, see Theorems 3.3 and 3.4. Section 4 is devoted to the construction of suitable barrier functions at infinity, which are then used in Sections 5 and 6 in the proofs of existence and uniqueness of bounded solutions of problems (1.1) and (1.2) with prescribed conditions at infinity. Finally, Section 7 contains some examples and applications of our main theorems

2. PRELIMINARIES

In this section we collect some notions and results from Riemannian Geometry following [7, 19].

Let (M, g) be a Riemannian manifold of dimension m with metric g . Let $p \in M$ and let (U, φ) be a *local chart* such that $p \in U$. Denote by x^1, \dots, x^m , $m = \dim M$, the coordinate functions on U . Then, at any $q \in U$ we have

$$(2.1) \quad g = g_{ij} dx^i dx^j$$

where dx^i denotes the differential of the function x^i and g_{ij} are the (local) components of the metric defined by $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$. Its inverse will be denoted by g^{ij} . In equation (2.1) and throughout this section we adopt the Einstein summation convention over repeated indices.

Note that the Laplacian of a function $u \in C^2(M)$ has locally the form

$$(2.2) \quad \Delta f = \mathfrak{g}^{-1} \frac{\partial}{\partial x_i} \left(\mathfrak{g} g^{ij} \frac{\partial f}{\partial x_j} \right),$$

where

$$\mathbf{g} = \sqrt{\det(\mathbf{g}_{ij})}.$$

Now, fix a point $o \in M$ and denote by $\text{Cut}(o)$ the *cut locus* of o . For any $x \in M \setminus [\text{Cut}(o) \cup \{o\}]$, one can define the *polar coordinates* with respect to o , see e.g. [7]. Namely, for any point $x \in M \setminus [\text{Cut}(o) \cup \{o\}]$ there correspond a polar radius $r(x) := \text{dist}(x, o)$ and a polar angle $\theta \in \mathbb{S}^{m-1}$ such that the shortest geodesics from o to x starts at o with direction θ in the tangent space $T_o M$. Since we can identify $T_o M$ with \mathbb{R}^m , θ can be regarded as a point of \mathbb{S}^{m-1} . For any $x_0 \in M$ and for any $R > 0$ we set $B_R(x_0) := \{x \in M : \text{dist}(x, x_0) < R\}$; in addition, we denote by $d\mu$ the Riemannian volume element on M , and by $S(x_0, R)$ the area of the sphere $\partial B_R(x_0)$.

The Riemannian metric in $M \setminus [\text{Cut}(o) \cup \{o\}]$ in polar coordinates reads

$$g = dr^2 + A_{ij}(r, \theta) d\theta^i d\theta^j,$$

where $(\theta^1, \dots, \theta^{m-1})$ are coordinates in \mathbb{S}^{m-1} and (A_{ij}) is a positive definite matrix. Let (A^{ij}) denote the inverse matrix of (A_{ij}) . It is not difficult to see that the Laplace-Beltrami operator in polar coordinates has the form

$$(2.3) \quad \Delta = \frac{\partial^2}{\partial r^2} + \mathcal{F}(r, \theta) \frac{\partial}{\partial r} + \Delta_{S_r},$$

where $\mathcal{F}(r, \theta) := \frac{\partial}{\partial r}(\log \sqrt{A(r, \theta)})$, $A(r, \theta) := \det(A_{ij}(r, \theta))$, Δ_{S_r} is the Laplace-Beltrami operator on the submanifold $S_r := \partial B_r(o) \setminus \text{Cut}(o)$.

M is a *manifold with a pole*, if it has a point $o \in M$ with $\text{Cut}(o) = \emptyset$. The point o is called *pole* and the polar coordinates (r, θ) are defined in $M \setminus \{o\}$.

A manifold with a pole is a *spherically symmetric manifold* or a *model*, if the Riemannian metric is given by

$$(2.4) \quad g = dr^2 + \psi^2(r) d\theta^2,$$

where $d\theta^2 = \beta_{ij} d\theta^i d\theta^j$ is the standard metric in \mathbb{S}^{m-1} , β_{ij} being smooth functions of $\theta^1, \dots, \theta^{m-1}$, and $\psi \in \mathcal{A}$, with being defined in (1.5). In this case, we write $M \equiv M_\psi$; furthermore, we have $\sqrt{A(r, \theta)} = \psi^{m-1}(r)$, so that

$$\Delta = \frac{\partial^2}{\partial r^2} + (m-1) \frac{\psi'}{\psi} \frac{\partial}{\partial r} + \frac{1}{\psi^2} \Delta_{\mathbb{S}^{m-1}},$$

where $\Delta_{\mathbb{S}^{m-1}}$ is the Laplace-Beltrami operator in \mathbb{S}^{m-1} . In addition, the boundary area of the geodesic sphere ∂S_R is computed by

$$S(o, R) = \omega_m \psi^{m-1}(R),$$

ω_m being the area of the unit sphere in \mathbb{R}^m . Also, the volume of the ball $B_R(o)$ is given by

$$\mu(B_R(o)) = \int_0^R S(o, \xi) d\xi.$$

Observe that for $\psi(r) = r$, $M = \mathbb{R}^m$, while for $\psi(r) = \sinh r$, M is the m -dimensional hyperbolic space \mathbb{H}^m .

Let us recall comparison results for sectional and Ricci curvatures that will be used in the sequel. Let $\text{Cut}^*(o) = \text{Cut}(o) \cup \{o\}$ and, for any $x \in M \setminus \text{Cut}^*(o)$, denote by $\text{Ric}_o(x)$ the *Ricci curvature* at x in the direction $\frac{\partial}{\partial r}$. Let ω denote any pair of tangent vectors from $T_x M$ having the form $(\frac{\partial}{\partial r}, X)$, where X is a unit vector orthogonal to $\frac{\partial}{\partial r}$. Denote by $K_\omega(x)$ the *sectional curvature* at the point x of the 2-section determined by ω . Observe that (see [7, Section 15], [11], [12]), if $\text{Cut}(o) = \emptyset$ and

$$(2.5) \quad K_\omega(x) \leq -\frac{\psi''(r)}{\psi(r)} \quad \text{for all } x = (r, \theta) \in M \setminus \{o\},$$

for some function $\psi \in \mathcal{A}$, then

$$(2.6) \quad \mathcal{F}(r, \theta) \geq (m-1) \frac{\psi'(r)}{\psi(r)} \quad \text{for all } r > 0, \theta \in \mathbb{S}^{m-1}.$$

On the other hand, if

$$(2.7) \quad \text{Ric}_o(x) \geq -(m-1) \frac{\phi''(r)}{\phi(r)} \quad \text{for all } x = (r, \theta) \in M \setminus \text{Cut}^*(o),$$

for some function $\phi \in \mathcal{A}$, then

$$(2.8) \quad \mathcal{F}(r, \theta) \leq (m-1) \frac{\phi'(r)}{\phi(r)} \quad \text{for all } r > 0, \theta \in \mathbb{S}^{m-1} \text{ with } x = (r, \theta) \in M \setminus \text{Cut}^*(o).$$

Note that if M_ψ is a model manifold, then for any $x = (r, \theta) \in M_\psi \setminus \{o\}$

$$\text{K}_\omega(x) = -\frac{\psi''(r)}{\psi(r)},$$

and

$$\text{Ric}_o(x) = -(m-1) \frac{\psi''(r)}{\psi(r)}.$$

Recall that a Riemannian manifold M is said to be *non-parabolic* if it admits a nonconstant positive superharmonic function, and *parabolic* otherwise (see e.g. [7]). Observe that M is non-parabolic if and only if it admits a positive Green function $G(x, y) < \infty$ for every $x, y \in M, x \neq y$; moreover,

$$\int_1^\infty \frac{d\xi}{S(o, \xi)} = \infty,$$

for some $o \in M$, if and only if M is parabolic (see [7, Theorem 7.5, Corollary 15.2]).

In the sequel, we also consider *Cartan-Hadamard* Riemannian manifolds, i.e. simply connected complete noncompact Riemannian manifolds with nonpositive sectional curvatures. Observe that (see, e.g. [7], [9]) on Cartan-Hadamard manifolds we have $\text{Cut}(o) = \emptyset$ for any $o \in M$.

3. EXISTENCE AND UNIQUENESS RESULTS

Before stating our main results, we need some preliminary materials. Concerning the coefficients of the operator \mathcal{L} , c and f we make the following set of assumptions:

$$(HP0) \quad \begin{cases} \text{(i)} & a \in C_{\text{loc}}^\sigma(M) \text{ for some } \sigma \in (0, 1), a > 0 \text{ in } M; \\ \text{(ii)} & c, f \in C_{\text{loc}}^\sigma(M) \cap L^\infty(M). \end{cases}$$

Note that the coefficient a can be unbounded at infinity.

For any $R > 0, \delta > 0, \theta_0 \in \mathbb{S}^{m-1}$ set

$$(3.1) \quad \mathcal{C}_{\theta_0, \delta}^R := \left\{ x \equiv (r, \theta) \in M : r > R, \text{dist}_{\mathbb{S}^{m-1}}(\theta, \theta_0) < \delta \right\},$$

where $\text{dist}_{\mathbb{S}^{m-1}}(\theta, \theta_0)$ denotes the geodesic distance on \mathbb{S}^{m-1} between θ and θ_0 .

Subsolutions, supersolutions and solutions of equation (1.1) and of problem (1.2) are meant as follows.

Definition 3.1. A function $u \in C^2(M)$ is a subsolution of equation (1.1) if

$$a\Delta u(x) + c(x)u(x) \geq f(x) \quad \text{for any } x \in M.$$

A supersolution is defined replacing the previous “ \geq ” with “ \leq ”. Finally, a solution is both a subsolution and a supersolution.

Definition 3.2. A function $u \in C_{x,t}^{2,1}(M \times (0, \infty)) \cap C(M \times [0, \infty))$ is a subsolution of problem (1.2) if

- (i) $\partial_t u(x, t) \leq a\Delta u(x, t) + c(x)u(x, t) + f(x) \quad \text{for any } x \in M, t \in (0, \infty),$
- (ii) $u(x, 0) \leq u_0(x) \quad \text{for any } x \in M.$

A supersolution is defined replacing the previous two “ \leq ” with “ \geq ”. Finally, a solution is both a subsolution and a supersolution.

In the following, the function

$$\omega(r) := \max_{i,j=1,\dots,m-1, \theta \in \mathbb{S}^{m-1}} \left\{ \left| \frac{\partial A^{ij}(r, \theta)}{\partial \theta^i} \right| + \frac{1}{2} \frac{|A^{ij}(r, \theta)|}{A(r, \theta)} \left| \frac{\partial A(r, \theta)}{\partial \theta^i} \right| + |A^{ij}(r, \theta)| \right\} \quad (r > 0)$$

will play an important role.

Our first result concerns the existence and uniqueness of solutions of elliptic equations with prescribed conditions at infinity.

Theorem 3.3. *Let assumptions (HP0)-(HP1) be satisfied. Let $\gamma \in C(\mathbb{S}^{m-1})$ and $c \leq 0$. Suppose that, for some $C_0 > 0$, $R_0 > 0$*

$$(3.2) \quad \frac{1}{\underline{a}(r)} \geq C_0 \omega(r) \quad \text{for all } x = (r, \theta) \in M, r \geq R_0.$$

Then there exists a unique solution of equation (1.1) such that condition (1.2) is satisfied.

Moreover, concerning the parabolic problem (1.2), we have the following result.

Theorem 3.4. *Let assumptions (HP0)-(HP1) be satisfied. Let $\tilde{\gamma} \in C(\mathbb{S}^{m-1} \times [0, \infty))$, and $u_0 \in C(M) \cap L^\infty(M)$. Suppose that conditions (1.8) and (3.2) are satisfied. Then there exists a unique solution of problem (1.2) such that condition (1.7) is satisfied.*

Remark 3.5. Note that if $M \equiv M_\psi$ is a model, then $\omega(r) = \frac{1}{\psi^2(r)}$. So, condition (3.2) reads as follows

$$(3.3) \quad \underline{a}(r) \leq \frac{\psi^2(r)}{C_0} \quad \text{for all } x = (r, \theta) \in M, r \geq R_0.$$

Remark 3.6. Note that if $\psi(r) = r$, and thus $M = \mathbb{R}^m$, conditions (HP1) and (3.3) cannot be simultaneously satisfied. Hence, our results cannot be applied.

4. CONSTRUCTION OF BARRIERS AT INFINITY

Lemma 4.1. *Let assumptions (HP0)-(HP1) be satisfied. Then there exists a supersolution V of equation*

$$(4.1) \quad a(x) \Delta V = -1 \quad \text{in } M,$$

such that

$$(4.2) \quad V(x) > 0 \quad \text{for all } x \in M,$$

and

$$(4.3) \quad \lim_{r(x) \rightarrow \infty} V(x) = 0.$$

Proof. Define

$$a_0(r) := \begin{cases} \frac{1}{C \underline{a}(R_0)} & \text{if } r \in [0, R_0) \\ \frac{1}{C \underline{a}(r)} & \text{if } r \in [R_0, \infty); \end{cases}$$

here

$$\bar{C} := \frac{1}{\underline{a}(R_0)} \min \left\{ \min a, \underline{a}(R_0) \right\} \in (0, 1].$$

Clearly, $a_0 \in C([0, \infty))$; moreover, by assumption (HP1)-(i) and by the definition of $a_0(r)$ and \bar{C} ,

$$(4.4) \quad a(x) \geq \frac{1}{a_0(r(x))} \quad \text{for every } x \in M.$$

Note that for every $r > 0$

$$(4.5) \quad \begin{aligned} & \left(\int_r^\infty \frac{d\xi}{\psi^{m-1}(\xi)} \right) \left(\int_0^r a_0(t) \psi^{m-1}(t) dt \right) - \int_0^r \left(\int_t^\infty \frac{d\xi}{\psi^{m-1}(\xi)} \right) a_0(t) \psi^{m-1}(t) dt \\ & \leq \int_r^\infty \frac{d\xi}{\psi^{m-1}(\xi)} \int_0^r \frac{\int_t^\infty \frac{d\xi}{\psi^{m-1}(\xi)}}{\int_r^\infty \frac{d\xi}{\psi^{m-1}(\xi)}} a_0(t) \psi^{m-1}(t) dt - \int_0^r \int_t^\infty \frac{d\xi}{\psi^{m-1}(\xi)} a_0(t) \psi^{m-1}(t) dt = 0. \end{aligned}$$

From (4.5) and hypothesis **(HP1)** we get

$$H := \limsup_{\rho \rightarrow \infty} \left\{ \int_\rho^\infty \frac{d\xi}{\psi^{m-1}(\xi)} \int_0^\rho a_0(t) \psi^{m-1}(t) dt - \int_0^\rho \int_t^\infty \frac{d\xi}{\psi^{m-1}(\xi)} a_0(t) \psi^{m-1}(t) dt \right\} \leq 0.$$

Define for every $x \in M$

$$(4.6) \quad \begin{aligned} V(x) & \equiv V(r(x)) \\ & := \left(\int_{r(x)}^\infty \frac{d\xi}{\psi^{m-1}(\xi)} \right) \left(\int_0^{r(x)} a_0(t) \psi^{m-1}(t) dt \right) - \int_0^{r(x)} \left(\int_t^\infty \frac{d\xi}{\psi^{m-1}(\xi)} \right) a_0(t) \psi^{m-1}(t) dt - H. \end{aligned}$$

We have that $V \in C^2(M)$. Furthermore, for every $r > 0$

$$(4.7) \quad V'(r) = -\frac{1}{\psi^{m-1}(r)} \int_0^r a_0(t) \psi^{m-1}(t) dt < 0,$$

$$(4.8) \quad V''(r) = (m-1) \frac{\psi'(r)}{\psi^m(r)} \int_0^r a_0(t) \psi^{m-1}(t) dt - a_0(r).$$

Then $V'(0) = 0$, $V''(0) = -\frac{a_0(0)}{m} < 0$ and in view of (4.8), **(HP1)**, (2.6), (2.3) and (4.4) we obtain

$$(4.9) \quad \begin{aligned} a(x) \Delta V(x) & = a(x) [V''(r) + \mathcal{F}(r, \theta) V'(r)] \\ & \leq a(x) \left[V''(r) + (m-1) \frac{\psi'(r)}{\psi(r)} V'(r) \right] \\ & \leq -a(x) a_0(r(x)) \leq -1 \quad \text{for all } x \in M. \end{aligned}$$

Finally, it is easily checked that (4.2) and (4.3) are satisfied. This completes the proof. \square

Lemma 4.2. *Let assumptions **(HP0)**-**(HP1)** be satisfied. Suppose that condition (3.2) holds. Let V be defined as in (4.6). Then there exist $\widehat{R} > 0$, $\widehat{\delta} > 0$, $\widehat{C} > 0$ such that for any $\theta_0 \in \mathbb{S}^{m-1}$ the function*

$$(4.10) \quad h(x; \theta_0) := \widehat{C} V(r) + \text{dist}_{\mathbb{S}^{m-1}}^2(\theta, \theta_0) \quad (x = (r, \theta) \in M)$$

is a supersolution of equation

$$(4.11) \quad a(x) \Delta h(\cdot; \theta_0) = -1 \quad \text{in } \mathcal{C}_{\theta_0, \widehat{\delta}}^{\widehat{R}}.$$

Moreover,

$$(4.12) \quad h(x; \theta_0) > 0 \quad \text{for all } x \in \overline{\mathcal{C}_{\theta_0, \widehat{\delta}}^{\widehat{R}}},$$

and for any $0 < \delta \leq \widehat{\delta}$, $R \geq \widehat{R}$ there exists $m_{\delta, R} > 0$ independent of θ_0 such that

$$(4.13) \quad h(x; \theta_0) \geq m_{\delta, R} > 0 \quad \text{for all } x \in \partial \mathcal{C}_{\theta_0, \delta}^R.$$

In addition,

$$(4.14) \quad \lim_{r \rightarrow \infty} h(r, \theta_0; \theta_0) = 0 \quad \text{uniformly w.r.t. } \theta_0 \in \mathbb{S}^{m-1}.$$

Proof. In view of (2.2) and (3.2), for each $r > 0, \theta_0 \in \mathbb{S}^{m-1}$ we have that

$$\begin{aligned}
 (4.15) \quad \Delta_{S_r} \text{dist}_{\mathbb{S}^{m-1}}^2(\theta, \theta_0) &= \frac{1}{\sqrt{A(r, \theta)}} \frac{\partial [\sqrt{A(r, \theta)} A^{ij}(r, \theta)]}{\partial \theta^i} \frac{\partial \text{dist}_{\mathbb{S}^{m-1}}^2(\theta, \theta_0)}{\partial \theta^j} + A^{ij}(r, \theta) \frac{\partial^2 \text{dist}_{\mathbb{S}^{m-1}}^2(\theta, \theta_0)}{\partial \theta^i \partial \theta^j} \\
 &\leq C \max_{i, j=1, \dots, m-1, \theta \in \mathbb{S}^{m-1}} \left\{ \left| \frac{\partial A^{ij}(r, \theta)}{\partial \theta^i} \right| + \frac{1}{2} \frac{|A^{ij}(r, \theta)|}{A(r, \theta)} \left| \frac{\partial A(r, \theta)}{\partial \theta^i} \right| + |A^{ij}(r, \theta)| \right\} \\
 &\leq \frac{C}{C_0} \frac{1}{\underline{a}(r)} \text{ whenever } \theta \in \mathbb{S}^{m-1}, \text{dist}_{\mathbb{S}^{m-1}}(\theta, \theta_0) < \hat{\delta},
 \end{aligned}$$

for some positive constants $C, \hat{\delta}$ independent of r, θ, θ_0 . From (4.4), (4.9), and (4.15) we deduce that

$$\begin{aligned}
 (4.16) \quad a(x) \Delta h(x; \theta_0) &\leq a(x) \left[-\hat{C} a_0(r(x)) + \frac{C}{C_0} \frac{1}{\underline{a}(r(x))} \right] \\
 &\leq a(x) \left(-\hat{C} + \frac{C}{C_0} \right) a_0(r(x)) \leq -1 \quad \text{for all } x \in \mathcal{C}_{\theta_0, \hat{\delta}}^{\hat{R}},
 \end{aligned}$$

provided $\hat{C} \geq \frac{C}{C_0} + 1$ and $\hat{R} > R_0$. Hence (4.11) has been shown. Finally, (4.12), (4.13), (4.14) follow by the very definition of h , with $m_\delta = \min\{\delta^2, \hat{C}V(R)\}$. \square

5. PROOF OF THEOREM 3.3

Proof of Theorem 3.3. By standard results (see, e.g. [6]), for any $j \in \mathbb{N}$ there exists a unique classical solution $u_j \in C^2(B_j) \cap C(\overline{B_j})$ of problem

$$(5.1) \quad \begin{cases} a \Delta u_j + c u_j = f & \text{in } B_j \\ u_j = \gamma_1 & \text{on } \partial B_j, \end{cases}$$

where

$$\gamma_1(x) \equiv \gamma_1(r, \theta) := \gamma(\theta) \quad \text{for all } r > 0, \theta \in \mathbb{S}^{m-1}.$$

Let V be the supersolution provided by Lemma 4.1 and consider $W = V + 1$. Since $W \geq 1$ on M , we have that for any $j \in \mathbb{N}$ the function

$$\bar{u} := \bar{C}W$$

is a supersolution to problem (5.1), provided

$$\bar{C} \geq \max\{\|f\|_\infty, \|\gamma\|_\infty\}.$$

Analogously, we have that for any $j \in \mathbb{N}$, the function $\underline{u} := -\bar{u}$ is a subsolution to the same problem. By the comparison principle, for any $j \in \mathbb{N}$,

$$(5.2) \quad |u_j| \leq \bar{C} \|W\|_\infty =: \tilde{C} \quad \text{in } B_j.$$

By usual compactness arguments (see, e.g., [6]), there exists a subsequence $\{u_{j_k}\} \subset \{u_j\}$ and a function $u \in C^2(M)$ such that

$$u := \lim_{k \rightarrow \infty} u_{j_k} \quad \text{in } M.$$

Moreover, u solves equations (1.1). In the sequel, we still denote by $\{u_j\}$ the sequence $\{u_{j_k}\}$.

Now, we show that (1.6) holds. In order to do this, fix any $\theta_0 \in \mathbb{S}^{m-1}$. For any $R > 0, \delta > 0, j > R$ define

$$N_\delta^j \equiv N_{\delta, \theta_0}^{j, R} := \mathcal{C}_{\theta_0, \delta}^R \cap B_j,$$

We shall prove some estimates in N_δ^j , by constructing suitable supersolutions and subsolutions to problem

$$(5.3) \quad \begin{cases} a \Delta v + c v = 0 & \text{in } N_{\delta, \theta_0}^{j, R} \\ v = 0 & \text{on } \partial N_{\delta, \theta_0}^{j, R}, \end{cases}$$

and then using the comparison principle.

Let \widehat{R} and $\widehat{\delta}$ be given by Lemma 4.2. Fix any $0 < \epsilon < 1$. Define, for some $K > 0$ to be fixed later,

$$\overline{v}(x) := Kh(x; \theta_0) + \gamma(\theta_0) + \epsilon - u_j(x) \quad (x \in \overline{N_\delta^j}),$$

where for every $\delta \in (0, \widehat{\delta}]$ we set $N_\delta^j \equiv N_{\delta, \theta_0}^{j, \widehat{R}}$. Note that there exists $0 < \delta = \delta(\epsilon) \leq \widehat{\delta}$ such that for all $j \in \mathbb{N}$, $x \equiv (r, \theta) \in \partial N_\delta^j \cap \partial B_j$

$$(5.4) \quad |u_j(x) - \gamma(\theta_0)| = |\gamma(\theta) - \gamma(\theta_0)| < \epsilon.$$

Observe that, since $\gamma \in C(\mathbb{S}^{m-1})$ and \mathbb{S}^{m-1} is compact, such a $\delta = \delta(\epsilon)$ does not depend on θ_0 . From (5.4) we obtain

$$(5.5) \quad \overline{v}(x) \geq 0 \quad \text{for all } x \in \partial N_\delta^j, r(x) = j.$$

From (5.2) and (4.13) we get that

$$(5.6) \quad \overline{v}(x) \geq Km_{\delta, \widehat{R}} + \gamma(\theta_0) + \epsilon - \widetilde{C} \geq 0 \quad \text{for all } x \in \partial N_\delta^j, \widehat{R} < r(x) < j,$$

choosing

$$(5.7) \quad K \geq \frac{\|\gamma\|_\infty + \widetilde{C}}{m_{\delta, \widehat{R}}},$$

where \widetilde{C} is defined by (5.2); hence, K also depends on $\delta(\epsilon)$. From (5.2) and (4.13) we can infer that, if (5.7) holds, then

$$(5.8) \quad \overline{v}(x) \geq 0 \quad \text{for all } x \in \partial N_\delta^j, r(x) = \widehat{R}.$$

Moreover, for all $x \in N_\delta^j$

$$(5.9) \quad a\Delta \overline{v} + c\overline{v} \leq -K + cKh + \|c\|_\infty(\|\gamma\|_\infty + 1) - f \leq 0,$$

for

$$(5.10) \quad K \geq \|c\|_\infty(\|\gamma\|_\infty + 1) + \|f\|_\infty.$$

Now, choose $\delta > 0$ so small that (5.4) holds and K so that (5.7) and (5.10) hold. Hence, from (5.5), (5.6), (5.8) and (5.9), the function \overline{v} is a supersolution of problem (5.3). So, by the comparison principle,

$$\overline{v} \geq 0 \quad \text{in } N_\delta^j.$$

Therefore,

$$(5.11) \quad u_j \leq Kh + \gamma(\theta_0) + \epsilon \quad \text{in } N_\delta^j.$$

Similarly we can show that, for the same $\delta = \delta(\epsilon)$ as in the previous calculations (see (5.4)), the function

$$\underline{v}(x) := -Kh(x; \theta_0) + \gamma(\theta_0) - \epsilon - u_j(x) \quad (x \in \overline{N_\delta^j})$$

is a subsolution to problem (5.3). Hence, by the comparison principle,

$$\underline{v} \leq 0 \quad \text{in } N_\delta^j.$$

Therefore,

$$(5.12) \quad u_j \geq -Kh + \gamma(\theta_0) - \epsilon \quad \text{in } N_\delta^j.$$

Letting $j \rightarrow \infty$ in (5.11) and (5.12) we obtain

$$-Kh(x; \theta_0) - \epsilon \leq u(x) - \gamma(\theta_0) \leq Kh(x; \theta_0) + \epsilon \quad \text{for any } x \equiv (r, \theta) \in \mathcal{C}_{\theta_0, \delta}^{\widehat{R}}.$$

In view of (4.14), taking $\theta = \theta_0$, letting $r \rightarrow \infty$, and $\epsilon \rightarrow 0^+$, we obtain that $u(r, \theta_0) \rightarrow \gamma(\theta_0)$ uniformly for $\theta_0 \in \mathbb{S}^{m-1}$.

It remains to prove the uniqueness of the solution. To do this, suppose, by contradiction, that there exist two solutions u_1 and u_2 of equation (1.1) satisfying condition (1.2). So, $w := u_1 - u_2$ solves equation

$$(5.13) \quad a\Delta w + cw = 0 \quad \text{in } M;$$

moreover,

$$(5.14) \quad \lim_{r(x) \rightarrow \infty} w(x) = 0.$$

Fix any $\epsilon > 0$. In view of (5.14), there exists $R_\epsilon > 0$ such that for any $R > R_\epsilon$

$$(5.15) \quad w \leq \epsilon \quad \text{on } \partial B_R.$$

Thus, u is a subsolution to problem

$$\begin{cases} a\Delta w + cw = 0 & \text{in } B_R \\ w = \epsilon & \text{on } \partial B_R. \end{cases}$$

By the comparison principle,

$$w \leq \epsilon \quad \text{in } B_R.$$

Letting $R \rightarrow \infty$ and $\epsilon \rightarrow 0^+$, we have

$$w \leq 0 \quad \text{in } M.$$

Similarly, it can be shown that $w \geq 0$ in M . So, $u_1 \equiv u_2$. This completes the proof of Theorem 3.3. \square

6. PROOF OF THEOREMS 3.4

Here and in the following, $\{\zeta_j\} \subset C_c^\infty(B_j)$ will be a sequence of functions such that, for each $j \in \mathbb{N}$, $0 \leq \zeta_j \leq 1$, $\zeta_j \equiv 1$ in $B_{j/2}$.

Proof of Theorem 3.4. Fix any $T > 0$. For any $j \in \mathbb{N}$ let $u_j \in C_{x,t}^{2,1}(B_j \times (0, T]) \cap C(\overline{B_j} \times [0, T])$ be the unique solution (see, e.g., [17]) of problem

$$(6.1) \quad \begin{cases} \partial_t u_j = a\Delta u_j + cu_j + f & \text{in } B_j \times (0, T], \\ u_j = \tilde{\gamma}_1 & \text{in } \partial B_j \times (0, T], \\ u_j = u_{0,j} & \text{in } B_j \times \{0\}, \end{cases}$$

where

$$\tilde{\gamma}_1(x, t) \equiv \tilde{\gamma}_1(r, \theta, t) =: \tilde{\gamma}(\theta, t) \quad \text{for all } r > 0, \theta \in \mathbb{S}^{m-1}, t \in [0, T];$$

furthermore,

$$(6.2) \quad u_{0,j}(x) := \zeta_j(x)u_0(x) + [1 - \zeta_j(x)]\tilde{\gamma}_1(x, 0) \quad \text{for all } x \in \overline{B_j}.$$

It is easily seen that the function

$$\bar{v}(x, t) := Ce^{\beta t} \quad ((x, t) \in M \times [0, T])$$

is a supersolution of problem (6.1) for any $j \in \mathbb{N}$, provided that

$$\beta \geq 1 + \|c\|_\infty, \quad C \geq \max\{\|f\|_\infty, \|\tilde{\gamma}\|_\infty, \|u_0\|_\infty\}.$$

Thus, by the comparison principle,

$$(6.3) \quad u_j(x, t) \leq \bar{v}(x, t) \quad \text{for all } (x, t) \in M \times [0, T].$$

Furthermore, the function

$$\underline{v}(x, t) := -Ce^{\beta t} \quad ((x, t) \in M \times [0, T])$$

is a subsolution of problem (6.1) for any $j \in \mathbb{N}$. Thus, by the comparison principle,

$$(6.4) \quad u_j(x, t) \geq \underline{v}(x, t) \quad \text{for all } (x, t) \in M \times [0, T].$$

From (6.3)-(6.4) we obtain

$$(6.5) \quad |u_j(x, t)| \leq Ce^{\beta T} =: K_T \quad \text{for all } (x, t) \in M \times [0, T].$$

By usual compactness arguments (see, e.g., [17]), there exists a subsequence $\{u_{j_k}\} \subseteq \{u_j\}$ which converges, as $k \rightarrow \infty$, to a solution $u \in C^2(M \times (0, T]) \cap C(M \times [0, T])$ of problem (1.2).

We claim that (1.7) holds. In fact, fix any $\theta_0 \in \mathbb{S}^{m-1}$, $t_0 \in [0, T]$, and $0 < \epsilon < 1$. Let \widehat{R} and $\widehat{\delta}$ be defined as in Lemma 4.2. Let

$$t_\delta := \max\{t_0 - \delta, 0\} \quad \text{for any } 0 < \delta \leq \widehat{\delta}.$$

Then there exists a positive constant $0 < \delta = \delta(\epsilon) < \widehat{\delta}$ such that

$$(6.6) \quad \widetilde{\gamma}(\theta_0, t_0) - \epsilon \leq \widetilde{\gamma}(\theta, t) \leq \widetilde{\gamma}(\theta_0, t_0) + \epsilon \quad \text{whenever } \text{dist}_{\mathbb{S}^{m-1}}(\theta, \theta_0) < \delta, t \in [t_\delta, t_0].$$

Note that $\widetilde{\gamma}$ is continuous in the compact set $\mathbb{S}^{m-1} \times [0, T]$, thus such a $\delta = \delta(\epsilon)$ does not depend on θ_0 and t_0 . Furthermore, due to (1.8), there exists $R_\epsilon > 0$ such that

$$(6.7) \quad \widetilde{\gamma}(\theta, 0) - \epsilon \leq u_0(r, \theta) \leq \widetilde{\gamma}(\theta, 0) + \epsilon \quad \text{for all } x = (r, \theta) \in M \setminus B_{R_\epsilon}.$$

Consider the function

$$(6.8) \quad \underline{w}(x, t) := -Kh(x; \theta_0)e^{\alpha t} - \lambda(t - t_0)^2 + \widetilde{\gamma}(\theta_0, t_0) - 3\epsilon, \quad (x, t) \in Q_\delta^j := N_\delta^j \times [t_\delta, t_0],$$

with $N_\delta^j = C_{\theta_0, \delta}^R \cap B_j$, where $R > \max\{\widehat{R}, R_\epsilon\}$ and where $K > 0$, $\alpha > 0$, $\lambda > 0$ are constants to be chosen later. We get

$$a\Delta \underline{w} + c\underline{w} \geq Ke^{\alpha t} - Kch(x; \theta_0)e^{\alpha t} - \|c\|_\infty(\|\widetilde{\gamma}\|_\infty + \lambda T^2 + 3) \quad \text{in } Q_\delta^j.$$

Therefore,

$$(6.9) \quad \begin{aligned} \partial_t \underline{w} - a\Delta \underline{w} - c\underline{w} - f &\leq -\alpha Kh(x, \theta_0)e^{\alpha t} - 2\lambda(t - t_0) - Ke^{\alpha t} + cKh(x, \theta_0)e^{\alpha t} \\ &\quad + \|c\|_\infty(\|\widetilde{\gamma}\|_\infty + \lambda T^2 + 3) + \|f\|_\infty \leq 0 \end{aligned} \quad \text{in } N_\delta^j \times (t_\delta, t_0),$$

if

$$(6.10) \quad \alpha \geq \|c\|_\infty,$$

and

$$(6.11) \quad K \geq 2\lambda T + \|f\|_\infty + \|c\|_\infty(\|\widetilde{\gamma}\|_\infty + \lambda T^2 + 3) + \|f\|_\infty.$$

Furthermore, it follows from (6.6) that for $j > R$

$$(6.12) \quad \underline{w}(x, t) \leq u_j(x, t) = \widetilde{\gamma}(\theta, t) \quad \text{for } x = (r, \theta) \in \partial N_\delta^j \cap \partial B_j, t \in (t_\delta, t_0).$$

Let $m_\epsilon = m_{\delta, R} > 0$ be the constant appearing in inequality (4.13), relative to $\partial C_{\theta_0, \delta}^R$ (recall that here $\delta = \delta(\epsilon)$ and $R = R(\epsilon)$). From (4.13) and (6.5) we can infer that

$$(6.13) \quad \underline{w}(x, t) \leq -Km_\epsilon + \|\widetilde{\gamma}\|_\infty \leq u_j(x, t) \quad \text{for all } x \in \partial C_{\theta_0, \delta}^R \cap B_j, t \in (t_\delta, t_0),$$

for

$$(6.14) \quad K \geq \frac{\|\widetilde{\gamma}\|_\infty + K_T}{m_\epsilon}.$$

Now, suppose that $t_\delta > 0$. From (6.5) we have that

$$(6.15) \quad \underline{w}(x, t_\delta) \leq u(x, t_\delta), \quad x \in N_\delta^j$$

if

$$(6.16) \quad \lambda \geq \frac{\|\tilde{\gamma}\|_\infty + K_T}{\delta^2}.$$

On the other hand, if $t_\delta = 0$ (this is always the case when $t_0 = 0$), then from (6.2), (6.6) and (6.7) we have that

$$(6.17) \quad \begin{aligned} \underline{w}(x, 0) &\leq \tilde{\gamma}(\theta_0, t_0) - 3\epsilon \leq \tilde{\gamma}(\theta_0, 0) - 2\epsilon \\ &\leq u_0(r, \theta) - \epsilon \leq u_{0,j}(r, \theta) = u_j(x, 0) \quad \text{for all } x = (r, \theta) \in \mathcal{C}_{\theta_0, \delta}^R \cap B_j. \end{aligned}$$

Now, suppose that (6.10), (6.11), (6.14) hold; moreover, assume (6.16), if $t_\delta > 0$. From (6.9), (6.12), (6.13), and (6.15) if $t_\delta > 0$ or (6.17) if $t_\delta = 0$, it follows that \underline{w} is a subsolution of problem

$$(6.18) \quad \begin{cases} \partial_t w = a\Delta w + cw + f & \text{in } Q_\delta^j \\ w = u_j & \text{in } \partial N_\delta^j \times (t_\delta, t_0], \\ w = u_j & \text{in } N_\delta^j \times \{t_\delta\}. \end{cases}$$

On the other hand, u_j is a solution of the same problem. Then by the maximum principle we have

$$(6.19) \quad \underline{w} \leq u_j \quad \text{in } Q_\delta^j.$$

Analogously we have that

$$(6.20) \quad u_j \leq \overline{w} \quad \text{in } Q_\delta^j,$$

where

$$(6.21) \quad \overline{w}(x, t) := Kh(x; \theta_0)e^{\alpha t} + \lambda(t - t_0)^2 + \tilde{\gamma}(\theta_0, t_0) + 3\epsilon, \quad (x, t) \in Q_\delta^j.$$

Finally, from (6.19) and (6.20) we have that for any $x \equiv (r, \theta) \in N_\delta^j$ and $t \in [t_\delta, t_0]$, with $0 < \epsilon < 1$, $j > R > \max\{\widehat{R}, R_\epsilon\}$ and $0 < \delta < \min\{\widehat{\delta}, \delta(\epsilon)\}$,

$$(6.22) \quad |u_j(x, t) - \tilde{\gamma}(\theta_0, t_0)| \leq Kh(x; \theta_0)e^{\alpha t} + l(t - t_0)^2 + 3\epsilon.$$

Note that these constants depend on ϵ , but do not depend on $\theta_0 \in \mathbb{S}^{m-1}$, $t_0 \in [0, T]$. Now we pass to the limit as $j \rightarrow \infty$ in (6.22), and choose $\theta = \theta_0$, $t = t_0$. So, for every $r > R$,

$$(6.23) \quad |u(r, \theta_0, t_0) - \tilde{\gamma}(\theta_0, t_0)| \leq Kh(r, \theta_0; \theta_0)e^{\alpha t_0} + 3\epsilon.$$

In view of (6.23) and (4.14), we have

$$|u(r, \theta_0, t_0) - \tilde{\gamma}(\theta_0, t_0)| < 4\epsilon$$

for $r > 0$ large enough, independent of $\theta_0 \in \mathbb{S}^{m-1}$, $t_0 \in [0, T]$. But $\epsilon > 0$ is arbitrarily small, therefore (1.7) follows.

In order to prove uniqueness, suppose by contradiction that there exist two solutions u_1, u_2 of problem (1.2) satisfying (1.7). Then set $w := u_1 - u_2$. Take any $\epsilon > 0$. In view of (1.7), there exists $R_\epsilon > 0$ such that

$$(6.24) \quad |w(x, t)| \leq \epsilon \quad \text{for all } x \in M \setminus B_{R_\epsilon}, t \in [0, T].$$

Moreover, w is a subsolution of problem

$$(6.25) \quad \begin{cases} \partial_t v = a\Delta v + cv & \text{in } B_{R_\epsilon} \times (0, T] \\ v = \epsilon & \text{in } \partial B_{R_\epsilon} \times (0, T] \\ v = 0 & \text{in } B_{R_\epsilon} \times \{0\}. \end{cases}$$

It is easily seen that the function

$$z(x, t) := \epsilon e^{\|c\|_\infty t} \quad (x \in M, t \in [0, T])$$

is a supersolution of problem (6.25). By the comparison principle,

$$(6.26) \quad w(x, t) \leq z(t) \leq \epsilon e^{\|c\|_\infty T} \quad \text{for all } x \in \overline{B}_{R_\epsilon}, t \in [0, T].$$

Similarly, it can be shown that

$$(6.27) \quad w(x, t) \geq -z(t) \geq -\epsilon e^{\|c\|_\infty T} \quad \text{for all } x \in \overline{B}_{R_\epsilon}, t \in [0, T].$$

Then, from (6.24), (6.26) and (6.27), we see that for some positive constant Λ we have

$$|w(x, y)| \leq \Lambda \epsilon \quad \text{for all } x \in M, t \in [0, T].$$

Letting $\epsilon \rightarrow 0^+$, we get $w \equiv 0$ in $M \times [0, T]$. Hence the proof is complete. \square

7. EXAMPLES

Example 7.1. Let $M = M_\psi$ be a model manifold with $\psi \in \mathcal{A}$, see (1.5), and let $\psi(r) \sim e^{r^\alpha}$ as $r \rightarrow \infty$ for some $\alpha > 0$. Note that for \mathbb{H}^m we have $\alpha = 1$. In general, there holds

$$\int_1^\infty \left(\int_r^\infty \frac{d\xi}{\psi^{m-1}(\xi)} \right) \frac{\psi^{m-1}(r)}{\underline{q}(r)} dr < \infty,$$

provided that

$$(7.1) \quad \int_1^\infty \frac{1}{r^{\alpha-1} \underline{q}(r)} dr < \infty.$$

Hence, if (7.1) holds, and

$$\underline{q}(r) \leq C_0 e^{2r^\alpha} \quad \text{for all } r \geq R_0$$

for some $R_0 > 0$, then Theorems 3.3 and 3.4 apply.

Moreover, note that in the special case when

$$a(r(x)) \equiv \underline{q}(r) \equiv 1,$$

condition (7.1) is satisfied for any $\alpha > 2$. Furthermore, observe that if $\psi = e^{r^\alpha}$ for all $r \geq r_0$, for some $r_0 > 0$, then we have

$$K_\omega(x) \sim -\alpha^2 [r(x)]^{2\alpha-2} \quad \text{as } r(x) \rightarrow \infty.$$

Example 7.2. Let M be a Cartan-Hadamard manifold. Suppose that, for some $\alpha > 0$,

$$(7.2) \quad K_\omega(x) \leq -\alpha^2 \quad \text{for all } x \in M.$$

Now note that by defining $\psi(r) := \frac{\sinh(\alpha r)}{\alpha}$ we have $\psi \in \mathcal{A}$ and $\frac{\psi''(r)}{\psi(r)} = \alpha^2$. Let $\omega(r)$ be defined as in (3.2) and suppose that

$$(7.3) \quad \int_1^\infty \omega(r) dr < \infty,$$

then **(HP1)**-(ii) and condition (3.2) hold, if we set $\underline{q}(r) := \frac{1}{\omega(r)}$.

Consider equation (1.1) with $c = f \equiv 0$, namely equation

$$(7.4) \quad a(x) \Delta u = 0 \quad \text{on } M,$$

which is of course equivalent to equation

$$(7.5) \quad \Delta u = 0 \quad \text{on } M.$$

Since $a(x)$ is arbitrary, we can choose $a \in C_{\text{loc}}^\sigma(M)$ for some $\sigma > 0$, with

$$(7.6) \quad a(x) \geq \underline{q}(r(x)) = \frac{1}{\omega(r(x))} \quad \text{for all } x \in M \setminus B_{R_0},$$

for some $R_0 > 0$. Thus, Theorem 3.3 applies for equation (7.4), and so also for equation (7.5). Moreover, Theorem 3.4 can be applied for the parabolic problem (1.2) with $c = f = 0$ and $a \in C_{\text{loc}}^\sigma(M)$ such that (7.6) is satisfied.

As already noted in Remark 3.5, if $M \equiv M_\psi$ is a model manifold then $\omega(r) = \frac{1}{\psi^2(r)}$. Clearly, (7.2) and (7.3) are satisfied on \mathbb{H}^m .

Example 7.3. Let $M \equiv M_\phi$ be a model manifold. Suppose that

$$(7.7) \quad K_\omega(x) \leq -\frac{A}{r^2 \log(r)} \quad \text{for all } x \in M \setminus B_{R_0},$$

for some $A > 1$, $R_0 > 0$. By [5, Proposition 3.4], for any $\beta \in (1, A)$ there exists some $R_1 \geq R_0$ such that

$$(7.8) \quad \phi(r) \geq \psi(r) \quad \text{for all } r \geq R_1,$$

where $\psi(r) := r \log^\beta(r)$. Moreover, for some $R_2 > 0$ large enough,

$$(7.9) \quad K_\omega(x) \leq -\frac{\psi''(r)}{\psi(r)} \quad \text{for all } x \in M \setminus B_{R_2}.$$

Now choose $a \in C_{\text{loc}}^\sigma(M)$, for some $\sigma > 0$, with

$$(7.10) \quad a(x) \geq \underline{a}(r(x)) = C_0 \psi^2(r(x)) \quad \text{for all } x \in M \setminus B_{R_2},$$

for some $C_0 > 0$. In view of (7.8), (7.9) and the very definition of ψ it is easily seen that hypothesis **(HP1)** and condition (3.2) are satisfied. Thus, Theorem 3.3 applies for equation (7.4), and hence also for equation (7.5). This is in accordance with [5, Theorem 3.6]. Moreover, Theorem 3.4 can be applied for the parabolic problem (1.6) with $c = f = 0$ and a defined as in (7.10).

REFERENCES

- [1] A. Ancona, Negatively curved manifolds, elliptic operators, and the Martin boundary, *Ann. Math.* **125** (1987), 495–536.
- [2] M.T. Anderson, The Dirichlet problem at infinity for manifolds of negative curvature, *J. Diff. Geom.* **18** (1983), 701–721.
- [3] M.T. Anderson, R. Shoen, Positive harmonic functions on complete manifolds of negative curvature, *Ann. Math.* **121** (1985), 429–461.
- [4] H. Brezis, S. Kamin, Sublinear elliptic equations in \mathbb{R}^N , *Manuscripta Math.* **74** (1992), 87–106.
- [5] H. Choi, Asymptotic Dirichlet problems for harmonic functions on Riemannian manifolds, *Trans. Amer. Math. Soc.* **281** (1984), 691–716.
- [6] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1983.
- [7] A. Grigoryan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, *Bull. Amer. Math. Soc.* **36** (1999), 135–249.
- [8] A. Grigor'yan, Heat kernels on weighted manifolds and applications. The ubiquitous heat kernel, 93–191, *Contemp. Math.*, 398, Amer. Math. Soc., Providence, RI, 2006.
- [9] A. Grigor'yan, “Heat Kernel and Analysis on Manifolds”, *AMS/IP Studies in Advanced Mathematics*, 47, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009.
- [10] G. Grillo, M. Muratori, F. Punzo, Conditions at infinity for the inhomogeneous filtration equation, *Ann. I. H. Poincaré-AN* **31** (2014), 413–428.
- [11] K. Ichihara, Curvature, geodesics and the Brownian motion on a Riemannian manifold. I. Recurrence properties, *Nagoya Math. J.* **87** (1982), 101–114.
- [12] K. Ichihara, Curvature, geodesics and the Brownian motion on a Riemannian manifold. II. Explosion properties, *Nagoya Math. J.* **87** (1982), 115–125.
- [13] A. M. Il'in, A. S. Kalashnikov, O. A. Oleinik, Linear equations of the second order of parabolic type, *Russian Math. Surveys* **17** (1962), 1–144.
- [14] S. Kamin, M.A. Pozio, A. Tesei, Admissible conditions for parabolic equations degenerating at infinity, *St. Petersburg Math. J.* **19** (2008), 239–251.
- [15] S. Kamin, F. Punzo, Prescribed conditions at infinity for parabolic equations, *Comm. Cont. Math.* **17** (2015), 1–19.
- [16] S. Kamin, F. Punzo, Dirichlet conditions at infinity for parabolic and elliptic equations, *Nonlin. Anal. (to appear)*.
- [17] O.A. Ladyzhenskaya, V.A. Solonnikov, N.A. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moscow (1967) (English translation: series Transl. Math. Monographs, **23** AMS, Providence, RI, 1968).

- [18] G. M. Lieberman, Second Order Parabolic Differential Equations, World Scientific, Singapore, 2005 .
- [19] P. Mastrolia, M. Rigoli, A. G. Setti, Yamabe-type equations on complete, noncompact manifolds, Progress in Mathematics, **302** (2012).
- [20] M. Murata, Non-uniqueness of the positive Cauchy problem for parabolic equations, J. Differential Equations, **123** (1995), 343–387.
- [21] M. Murata, Uniqueness and non-uniqueness of the positive Cauchy problem for the heat equation on Riemannian manifolds, Proc. Amer. Math. Soc., **123** (1995), 1923–1932.
- [22] Y. Pinchover, On the equivalence of Green functions of second order elliptic equations in \mathbb{R}^n , Diff. Int. Eq. **5** (1992), 481–493 .
- [23] Y. Pinchover, On uniqueness and nonuniqueness of the positive Cauchy problem for parabolic equations with unbounded coefficients, Math. Z. **223** (1996), 569–586 .
- [24] M.A. Pozio, F. Punzo, A. Tesei, Uniqueness and nonuniqueness of solutions to parabolic problems with singular coefficients, Discr. Cont. Dyn. Syst.-A **30** (2011), 891–916 .
- [25] F. Punzo, On the Cauchy problem for nonlinear parabolic equations with variable density, J. Evol. Eq. **9** (2009), 429–447.
- [26] F. Punzo, A. Tesei, Uniqueness of solutions to degenerate elliptic problems with unbounded coefficients, Ann. Inst. H. Poincaré (C) AN, **26** (2009), 2001–2024 .
- [27] D. Sullivan, The Dirichlet problem at infinity for a negatively curved manifold, J. Diff. Geom. **18** (1983), 723–732 .

Paolo Mastrolia
 Università degli Studi di Milano
 Dip di Matematica
 via Saldini 50
 20133 Milano.

Dario D. Monticelli
 Politecnico di Milano
 Dip. di Matematica
 via Bonardi 9
 20133 Milano.

Fabio Punzo
 Università della Calabria
 Dip. di Matematica e Informatica
 via Bucci, Cubo 31B
 87036 Rende (CS).